

# On the weak Kähler-Ricci flow

X. X. Chen, G. Tian and Z. Zhang \*

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Weak solutions of Kähler-Ricci flow</b>	<b>3</b>
<b>3</b>	<b>Higher order estimates</b>	<b>11</b>
3.1	The $C^{1,1}$ case . . . . .	12
3.2	The $L^\infty$ case . . . . .	13
<b>4</b>	<b><math>C^{1,1}</math> K-energy minimizer</b>	<b>15</b>

## 1 Introduction

Let  $M$  be an  $n$ -dimensional compact (without boundary) Kähler manifold. A Kähler metric  $g$  can be given by its Kähler form  $\omega_g$  on  $M$ . Consider the Kähler-Ricci flow on  $M \times [0, T)$

$$\frac{\partial \omega_{g(t)}}{\partial t} = -\text{Ric}(\omega_{g(t)}), \quad (1.1)$$

where  $g(t)$  is a family of Kähler metrics and  $\text{Ric}(\omega_g)$  denotes the Ricci curvature of  $g$ . It is known that for any smooth Kähler metric  $g_0$ , there is a unique solution  $g(t)$  of (1.1) for some maximal time  $T > 0$  with  $g(0) = g_0$ . In general,  $T$  will depend on the initial metric  $g_0$ . However, in Kähler manifold, this only depends on the Kähler class and the first Chern class. This observation plays a important role in our introduction of weak Ricci flow in Kähler manifold.

The Ricci flow was introduced by R. Hamilton in [18]. Extensive research has been done in the case of the smooth flow (c.f. [20][5] [13][19] [23] [11] [12] [28] or [14] for complete updated references). In order to prove the uniqueness of extremal Kähler metrics in full generality, in

---

\*The first two authors are partially supported by NSF funds.

[8], we were led to the study of Kähler-Ricci flow in the weak sense. More precisely, we proved in [8] that for any initial  $L^\infty$ -bounded Kähler metric, there is a uniformly  $L^\infty$ -bounded solution of (1.1) with the given initial “metric” in a suitable sense. Moreover, the volume form of the solution converges strongly to the volume form of the initial Kähler “metric” in  $L^2$ -topology as  $t \rightarrow 0$ .

As one may expect, such a weak solution of (1.1) should become smooth immediately after  $t > 0$ . Indeed, we confirm this in this note in a more general setting.

**Theorem 1.1.** *For any Kähler current  $g_0$  with  $C^{1,1}$  bounded potential on  $M$ , there is a unique smooth solution  $g(t)$  ( $t \in (0, T)$ ) of (1.1) such that  $\lim_{t \rightarrow 0^+} g(t) = g_0$  in  $C^{1,\alpha}(\forall \alpha \in (0, 1))$  norm at the potential level.*

A special case of Riemann surfaces was considered also in [10] with completely different proof.

It was known that any Kähler metric with constant scalar curvature is the absolute minimizer of the K-energy on the space of all Kähler metrics with a fixed Kähler class ([7], [16], [22] [8])<sup>1</sup>. From analytic point of view, it is an extremely difficult problem to prove the existence of Kähler metric with constant scalar curvature in general cases. One approach is to construct weak minimizers of the K-energy by applying certain variational or continuous methods. This seems a plausible direct approach but very hard. However, even if we obtained a  $C^{1,1}$  minimizer of the K energy functional, we would still face the regularity problem. In [7], the first named author made the following conjecture: *Any  $C^{1,1}$  minimizers of the K-energy in a given Kähler class must be smooth.* As a consequence of the above theorem, we can solve this conjecture in canonically polarized cases.

**Corollary 1.2.** *In a Kähler class which is proportional to the first Chern class, any  $L^\infty$  Kähler metric which minimizes the K-energy functional must be smooth.*

There is another motivation for this short paper. On a general Kähler manifold, the Kähler-Ricci flow (1.1) may develop singularity at finite time (see [29]). In an on-going project with his collaborators, the second named author proposed some problems of studying how the Kähler-Ricci flow extends across the finite time singularity (see [26] for more discussions). One of them involves constructing solutions of the Kähler-Ricci flow with much weaker initial metrics, possibly on spaces with mild singularity. Our second result gives a partial solution to this problem.

First we recall some standard facts: by Hodge Theorem, the space of all Kähler metrics in a fixed Kähler class given by  $\omega$  is

$$P(M, \omega) = \{\varphi \in C^\infty(M) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}.$$

---

<sup>1</sup> For Kähler Einstein metrics, cf. [2] [15] [27].

We denote by  $Cl_{L^\infty}P(M, \omega)$  the closure of  $P(M, \omega)$  in the space of all bounded functions in the  $L^\infty$ -topology. For any  $\varphi \in Cl_{L^\infty}P(M, \omega)$ , there is a well-defined volume form  $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n > 0$  in the weak sense. Now we can state our second result.

**Theorem 1.3.** *For any Kähler potential  $\varphi_0 \in Cl_{L^\infty}P(M, \omega)$  whose volume form is  $L^p(M, \omega)$  ( $p \geq 3$ ), there is a unique smooth solution  $g(t)$  of (1.1) for  $t \in (0, T)$  such that  $\frac{\omega_{g(t)}^n}{\omega^n}$  converges to  $\frac{\omega_{\varphi_0}^n}{\omega^n}$  strongly in the  $L^2$ -topology.*

The organization of this note is as follows: In section 2, we show the existence of the weak solutions for the induced potential flow which is equivalent to the Kähler-Ricci flow. We derive the  $C^0$ -estimates for those solutions and their time derivative. We also examine convergence problem for these weak solutions as  $t$  tends to 0. In section 3, we derive the 2nd and 3rd order estimates for the weak solutions. Then Theorem 1.1 and 1.3 follow. In last section, we prove the corollary.

## 2 Weak solutions of Kähler-Ricci flow

First we reduce the Kähler-Ricci flow (1.1) to a scalar flow. To start with, we note that  $[\omega_{g_0}] - tc_1(M)$  represents a Kähler class whenever  $0 \leq t \leq T$  for a sufficiently small  $T > 0$  depending only on  $[\omega_{g_0}]$  and  $C_1(M)$ . Choose a smooth family of Kähler forms  $\omega_t$  for  $t \in [0, T]$  such that  $\omega_0 = \omega_{g_0}$  and  $[\omega_t] = [\omega_{g_0}] - tc_1(M)$ . Write  $\omega_{g(t)} = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ . Then, one can show by standard arguments that  $g(t)$  solves (1.1) if and only if  $\varphi(t)$  solves the following scalar flow:

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_t^n} - h_{\omega_t}, \quad (2.1)$$

where  $h_{\omega_t}$  is defined by

$$\text{Ric}(\omega_t) + \frac{\partial \omega_t}{\partial t} = \sqrt{-1}\partial\bar{\partial}h_{\omega_t}, \text{ and } \int_X (e^{h_{\omega_t}} - 1)\omega_t^n = 0.$$

Clearly such a  $h(\omega_t)$  does exist and is unique. As usual, we will regard either (1.1) or (2.1) as the Kähler-Ricci flow on  $M$ . For simplicity, we denote by  $\omega_\varphi$  the Kähler form  $\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$ .

To construct a weak solution to the Kähler-Ricci flow equation with non-smooth initial Kähler potentials, as usual, we use smooth approximations of the initial data. Let  $\varphi_0$  be a bounded Kähler potential such that its volume form is in  $L^p$  for some  $p > 1$ . Let  $\varphi_0(s)$  ( $0 \leq s \leq 1$ ) be a 1-parameter Kähler potentials such that the following properties hold:

1.  $\varphi_0(0) = \varphi_0$  and  $\varphi_0(s)$  ( $0 < s \leq 1$ )  $\in P(M, \omega)$ .
2. If  $\varphi_0$  has  $C^{1,1}$  bound, then  $\varphi_0(s)$  has uniform  $C^{1,1}$  upper bound and  $\varphi_0(s)$  ( $s > 0$ )  $\rightarrow \varphi_0$  strongly in  $W^{2,q}(M, \omega)$  for  $q$  sufficiently large.

3.  $\varphi_0(s)$  converges to  $\varphi_0$  uniformly in  $L^\infty$ -topology and the volume form ratio converges to  $\frac{\omega_{\varphi_0}^n}{\omega^n}$  strongly in  $L^p(p > 1)$ .

It is known (cf. [29]) that for any  $\varphi$  in  $P(M, \omega)$ , there is a unique smooth solution of (2.1) on  $[0, T]$  with  $\varphi$  as the initial value. Therefore, we have  $\varphi(s, t) \in P(M, \omega)$  ( $0 < s \leq 1, t \in [0, T]$ ) satisfying:

$$\frac{\partial \varphi(s, t)}{\partial t} = \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_t^n} - h_{\omega_t}, \quad (2.2)$$

$$\varphi(s, 0) = \varphi_0(s). \quad (2.3)$$

Clearly, for each  $s > 0$  fixed, there exists a uniform  $C^{2, \alpha}$  bound for  $\varphi(s, t)$  ( $0 \leq t \leq T$ ). However, the upper bound may well depend on  $s$  and so may blow up when  $t, s$  are both small. If there is a limit

$$\varphi(t) = \lim_{s \rightarrow 0} \varphi(s, t), \quad \forall t \in [0, T],$$

then we can regard  $\varphi(t)$  as a weak solution of (2.1) with initiated value  $\varphi_0$ .

**Lemma 2.1.** *The solutions  $\varphi(s, t)$  converges uniformly to a family of functions  $\varphi(t)$  ( $t \in [0, T]$ ) as  $s$  tends to 0.*

*Proof.* For any positive  $s$  and  $s'$ , put  $\psi(t) = \varphi(s', t) - \varphi(s, t)$ , then we have

$$\frac{\partial \psi}{\partial t} = \log \frac{(\omega_{\varphi(s, t)} + \sqrt{-1} \partial \bar{\partial} \psi)^n}{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(s, t))^n}.$$

Then the Maximum principle implies that  $\sup_M |\psi(t)| \leq \sup_M |\psi(0)|$ , that is,

$$\sup_M |\varphi(s', t) - \varphi(s, t)| \leq \sup_M |\varphi_0(s') - \varphi_0(s)|.$$

The lemma follows easily from our choices of  $\varphi_0(s)$  and standard arguments on uniform convergence.  $\square$

We will show that  $\varphi(t)$  solves the Kähler-Ricci flow and consequently a weak solution in a suitable sense. For this purpose, we need to show that  $\varphi(t)$  is a smooth family for  $t > 0$  which solves (2.1) for  $t > 0$  such that  $\lim_{t \rightarrow 0+} \varphi(t) = \varphi_0$ . Of course, this is the core part of this paper. First we derive a few prior estimates on  $\varphi(s, t)$  ( Sometimes we abbreviated  $\varphi(s, t)$  as  $\varphi$  for simplicity). Clearly, these estimates pass to  $\varphi(t)$  by taking limits as  $s$  tends to 0.

**Lemma 2.2.** *If  $\varphi_0$  is bounded, then there exists a uniform constant  $C$  such that for any  $s \in (0, 1]$  and  $t \in [0, T]$ ,*

$$|\varphi(s, t)| \leq C.$$

*Proof.* Choose  $c$  such that  $|h_{\omega_t}|_{C^0} \leq c$  for all  $t \in [0, T]$ . Applying the Maximum Principle to  $\varphi(s, t) \pm ct$ , we can obtain

$$\min \varphi_0(s) - ct \leq \varphi(s, t) \leq \max \varphi_0(s) + ct, \quad \forall s > 0.$$

Then the bound on  $\varphi$  follows.  $\square$

The next lemma gives an estimate on the volume form for the Kähler-Ricci flow when  $t > 0$ .

**Lemma 2.3.** *For  $t > 0$ , the volume form has uniform upper and positive lower bounds which may depend on  $t$ .*

*Proof.* Differentiating the Kähler-Ricci flow equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_t^n} - h_{\omega_t},$$

we get

$$\begin{aligned} \varphi'' &= \Delta \varphi' - h'_{\omega_t} - \text{tr}_{\omega_t} \left( \frac{\partial}{\partial t} \omega_t \right) + \text{tr}_{\varphi} \left( \frac{\partial}{\partial t} \omega_t \right) \\ &\leq \Delta \varphi' + C(1 + \text{tr}_{\varphi} \omega). \end{aligned}$$

Similarly, we have

$$\varphi'' \geq \Delta \varphi' - C(1 + \text{tr}_{\varphi} \omega).$$

Consider

$$F_+ = -\varphi + t\varphi'.$$

Then,

$$\begin{aligned} \frac{\partial}{\partial t} F_+ &= -\varphi' + \varphi' + t\varphi'' \\ &\leq t\Delta \varphi' + Ct(1 + \text{tr}_{\varphi} \omega) \\ &= \Delta(-\varphi + t\varphi') + \Delta_{\varphi} \varphi + Ct(1 + \text{tr}_{\varphi} \omega) \\ &= \Delta_{\varphi} F_+ + n - g_{\varphi}^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} + Ct(1 + \text{tr}_{\varphi} \omega) \\ &\leq \Delta_{\varphi} F_+ + n + 1 - (1 - Ct)(1 + \text{tr}_{\varphi} \omega). \end{aligned}$$

For  $t$  small enough, we have

$$\frac{\partial}{\partial t} F_+ \leq \Delta_{\varphi} F_+ + n + 1.$$

Since  $F_+(0)$  is uniformly bounded from above, it follows from the Maximum principle that  $F_+$  has a uniform upper bound.

For the lower bound, set

$$F_- = \varphi + t\varphi' = \varphi - t h_{\omega} + t \log \frac{\omega_{\varphi}^n}{\omega_t^n}.$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial t} F_- &= \varphi' + \varphi' + t\varphi'' \\
&\geq 2\varphi' + t(\Delta\varphi' - C(1 + \text{tr}_\varphi\omega)) \\
&= \Delta(\varphi + t\varphi') - \Delta_\varphi\varphi + 2\varphi' - Ct(1 + \text{tr}_\varphi\omega) \\
&= \Delta_\varphi F_- - n + \text{tr}_\varphi\omega + 2\log \frac{\omega_\varphi^n}{\omega_t^n} - 2h_{\omega_t} - Ct(1 + \text{tr}_\varphi\omega) \\
&\geq \Delta_\varphi F_- - n - 1 + (1 - Ct)(1 + \text{tr}_\varphi\omega) + 2\log \frac{\omega_\varphi^n}{\omega_t^n} \\
&\geq \Delta_\varphi F_- - n - 1 + (1 - Ct) \left( \frac{\omega_\varphi^n}{\omega_t^n} \right)^{-\frac{1}{n}} + 2\log \frac{\omega_\varphi^n}{\omega_t^n}.
\end{aligned}$$

An easy computation shows

$$x^{\frac{-1}{n}} + 4\log x \geq 4n(1 - \log(4n)).$$

It follows that if  $2tC \leq 1$ , we have

$$\frac{\partial}{\partial t} F_- \geq \Delta_\varphi F_- - 2n \log(4n).$$

Then the Maximum principle implies that  $F_- > -C$  for some uniform constant  $C$ , that is, the volume form  $\frac{\omega_\varphi^n}{\omega_t^n}$  has a uniform positive lower bound after  $t > 0$ .

To prove the lemma for all  $t > 0$ , we can simply replace  $t$  by  $t - t_0$  and repeat the above arguments.  $\square$

Next we exam if  $\lim_{t \rightarrow 0+} \varphi(t) = \varphi_0$ . To make sure this, we need to have some integral estimates.

**Lemma 2.4.** *If  $\varphi_0$  is bounded and  $\frac{\omega_{\varphi_0}^n}{\omega^n}$  is in  $L^p(p \geq 1)$ , then for any  $s \in (0, 1]$ , there exists a positive function  $C(t)$  of  $t$  such that*

$$\left\| \frac{\omega_\varphi^n}{\omega^n} \right\|_{L^p} \leq C(t).$$

Moreover,  $C(t)$  is independent of  $s$  and is bounded for any finite time  $t$ .

*Proof.* For any sufficiently large  $\lambda > 0$ , set the modified volume ratio as

$$f_\varphi = \frac{\omega_\varphi^n}{\omega_t^n} e^{-\lambda\varphi}.$$

First we assume that  $\varphi_0$  is smooth. Let  $C$  be a constant satisfying:

$$\left| \frac{\partial}{\partial t} \omega_t \right|_{\omega_t} + |\text{Ric}(\omega_t)|_{\omega_t} + |\text{Ric}(\omega)|_{\omega_t} + |\partial\bar{\partial}h_{\omega_t}|_{\omega_t} \leq C.$$

Then,

$$\begin{aligned}
\frac{\partial \log f_\varphi}{\partial t} &= \operatorname{tr}_\varphi \left( \frac{\partial}{\partial t} \omega_t \right) + \Delta_\varphi \varphi' - \operatorname{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t - \lambda \varphi' \\
&= \Delta_\varphi \log f_\varphi + \operatorname{tr}_\varphi \left( \frac{\partial}{\partial t} \omega_t + \lambda \partial \bar{\partial} \varphi - \partial \bar{\partial} h_{\omega_t} \right) + R_{\omega_t} - \operatorname{tr}_{\omega_t} h_{\omega_t} - \lambda \varphi' \\
&= \Delta_\varphi \log f_\varphi + \lambda - (\lambda - 2C) \cdot \operatorname{tr}_\varphi \omega - \lambda \log f_\varphi + \lambda h_{\omega_t} + 2C \\
&\leq \frac{\Delta_\varphi f_\varphi}{f_\varphi} - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi^2} + 2C - \lambda \log f_\varphi.
\end{aligned}$$

Here we have used  $\lambda > C$ . Thus, we have

$$\frac{\partial f_\varphi}{\partial t} \leq \Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} + 2C f_\varphi - \lambda f_\varphi \log f_\varphi.$$

For any  $p \geq 1$ , we have

$$\begin{aligned}
&\frac{d}{dt} \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\
&= p \int_M f_\varphi^{p-1} \frac{\partial f_\varphi}{\partial t} e^{\lambda \varphi} \omega_t^n + \int_M f_\varphi^p e^{\lambda \varphi} \frac{\partial \omega_t^n}{\partial t} + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \frac{\partial \varphi}{\partial t} \omega_t^n \\
&= p \int_M f_\varphi^{p-1} \frac{\partial f_\varphi}{\partial t} e^{\lambda \varphi} \omega_t^n + \int_M f_\varphi^p (-R(\omega_t) + \operatorname{tr}_{\omega_t} h_{\omega_t}) e^{\lambda \varphi} \omega_t^n \\
&\quad + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \left( \log \frac{\omega_\varphi^n}{\omega_t^n} - h_{\omega_t} \right) \omega_t^n \\
&\leq p \int_M f_\varphi^{p-1} \left( \Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} + f_\varphi (2C - \lambda \log f_\varphi) \right) e^{\lambda \varphi} \omega_t^n + C \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\
&\quad + \lambda \int_M f_\varphi^p e^{\lambda \varphi} \left( \log \frac{\omega_\varphi^n}{\omega_t^n} - h_{\omega_t} \right) \omega_t^n \\
&\leq \int_M f_\varphi^{p-2} \left( p \left( \Delta_\varphi f_\varphi - \frac{|\nabla f_\varphi|_\varphi^2}{f_\varphi} \right) + f_\varphi (2pC - \lambda(p-1) \log f_\varphi) \right) \omega_\varphi^n + C_\lambda \int_M f_\varphi^p e^{\lambda \varphi} \omega_t^n \\
&\leq -p(p-1) \int_M f_\varphi^{p-2} |\nabla f_\varphi|_\varphi^2 e^{\lambda \varphi} \omega_t^n + C_\lambda \int_M (f_\varphi^p + f_\varphi^{p-1}) e^{\lambda \varphi} \omega_t^n.
\end{aligned}$$

Here we have used the fact that the function  $x \log x$  has a lower bound for  $x > 0$ .

Since  $\varphi$  is uniformly bounded, it follows that  $\frac{\omega_\varphi^n}{\omega^n}$  is uniformly bounded in  $L^p$  for all  $t \in [0, T]$ .

For general  $\varphi_0$ , we use above approximations  $\varphi_0(s)$ , then we have uniform estimates for  $\frac{\omega_{\varphi(s,t)}^n}{\omega^n}$  and consequently,  $\frac{\omega_\varphi^n}{\omega^n}$  is uniformly bounded in  $L^p$  for all  $t \in [0, T]$  by taking the limits.  $\square$

It follows from this lemma that the  $L^p$ -norm of  $\varphi(t)$  is uniformly bounded. By the work of Kolodziej [25], we know that  $\|\varphi(t)\|_{C^\alpha}$  is uniformly bounded, where  $\alpha = \alpha(p) > 0$  may depend on  $p > 1$ . Then for any sequence  $t_i \rightarrow 0$ , there is a subsequence, still denoted by  $t_i$  for simplicity, such that  $\varphi(t_i)$  converges to a  $C^\alpha$ -function  $\tilde{\varphi}_0$  in the  $C^\beta$ -topology for some  $\beta \in (0, \alpha)$ .<sup>2</sup> *A priori*, this limit potential might depend on the sequence we choose.

---

<sup>2</sup>If Lemma 2.4 implies that the  $L^p$  norm varies continuously, then  $\beta$  can be taken to be  $\alpha$  by the work of Kolodziej.

**Lemma 2.5.** *If  $p \geq 3$ , we have  $\tilde{\varphi}_0 = \varphi_0$ . Moreover,  $\frac{\omega_{\varphi(t)}^n}{\omega^n}$  converges to  $\frac{\omega_{\varphi_0}^n}{\omega^n}$  strongly in the  $L^2$ -topology.*

*Proof.* First we assume that  $\varphi_0$  is smooth, say one of  $\varphi_0(s)$  ( $s > 0$ ). In the preceding lemma, set  $p = 3$ , then we have

$$\frac{d}{dt} \int_M f_\varphi^3 e^{\lambda\varphi} \omega_t^n \leq -6 \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 e^{\lambda\varphi} \omega_t^n + C \int_M f_\varphi^3 e^{\lambda\varphi} \omega_t^n + C.$$

It implies

$$\int_0^t \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 \omega_t^n du \leq C. \quad (2.4)$$

Now set

$$f = \frac{\omega_\varphi^n}{\omega_t^n}.$$

**Claim 1:** We have

$$\int_0^t \int_M |\nabla f|_\varphi^2 \omega_\varphi^n du = \int_0^t \int_M f |\nabla f|_\varphi^2 \omega_t^n du \leq C \quad (2.5)$$

and

$$\int_0^t \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n du \leq C. \quad (2.6)$$

**Proof of Claim 1:** Since  $\varphi$  is uniformly bounded, we can choose a uniform constant  $c$  such that  $\varphi - c \leq 0$ . Then we have

$$\begin{aligned} & \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \\ &= -2 \int_M f_\varphi (\nabla f_\varphi \cdot \nabla \varphi)_\varphi (\varphi - c) \omega_\varphi^n - \int_M f_\varphi^2 (\varphi - c) \Delta_\varphi \varphi \omega_\varphi^n \\ &\leq \frac{1}{2} \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n + C \int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n - n \int_M f_\varphi^2 (\varphi - c) \omega_\varphi^n + \int_M f_\varphi^2 (\varphi - c) \operatorname{tr}_\varphi \omega_t \omega_\varphi^n \\ &\leq \frac{1}{2} \int_M f_\varphi |\nabla \varphi|_\varphi^2 \omega_t^n + C \int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n + C \int_M f_\varphi^2 \omega_\varphi^n \\ &= \frac{1}{2} \int_M f_\varphi |\nabla \varphi|_\varphi^2 \omega_t^n + C \int_M f_\varphi |\nabla f_\varphi|_\varphi^2 \omega_t^n + C \int_M f_\varphi^3 e^{\lambda\varphi} \omega_t^n. \end{aligned}$$

It follows

$$\int_0^t \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n du \leq C.$$

To prove the first inequality, we observe

$$f = e^{\lambda\varphi} f_\varphi.$$

Then,

$$\begin{aligned} \int_M f |\nabla f|_\varphi^2 \omega_t^n &= \int_M |\nabla(f_\varphi e^{\lambda\varphi})|_\varphi^2 e^{\lambda\varphi} \omega_\varphi^n \\ &\leq 2 \int_M |\nabla f_\varphi|^2 e^{3\lambda\varphi} \omega_\varphi^n + 2\lambda^2 \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 e^{3\lambda\varphi} \omega_\varphi^n \\ &\leq C \left( \int_M |\nabla f_\varphi|_\varphi^2 \omega_\varphi^n + \int_M f_\varphi^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \right). \end{aligned}$$



Then **Claim 1** follows easily.

**Claim 2:** For any smooth non-negative cut-off function  $\chi$  (fixed), we have,

$$\int_M f |\nabla \chi|_\varphi^2 \omega_t^n = \int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n \leq C(|\nabla \chi|_{L^\infty}). \quad (2.7)$$

**Proof of Claim 2:**

$$\begin{aligned} \int_M f |\nabla \chi|_\varphi^2 \omega_t^n &= \int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n \\ &\leq \int_M |\nabla \chi|_{\omega_t}^2 \operatorname{tr}_\varphi \omega_t \omega_\varphi^n \\ &\leq C \int_M \operatorname{tr}_\varphi (\omega_\varphi - \partial \bar{\partial} \varphi) \omega_\varphi^n \\ &= C \int_M (n - \Delta_\varphi \varphi) \omega_\varphi^n \leq C, \end{aligned}$$

where the first inequality follows from an elementary inequality and the second inequality makes use of the positivity of  $\omega_t$ .

In other words, any smooth cut-off function is automatically in  $W^{1,2}$  with respect to any Kähler metric in any given Kähler class.

**Claim 3:** We have

$$\int_M f |\nabla \varphi|_\varphi^2 \omega_t^n = \int_M |\nabla \varphi|_\varphi^2 \omega_\varphi^n \leq C(|\varphi|_{L^\infty}). \quad (2.8)$$

**Proof of Claim 3:** Let  $c$  be given in the proof of **Claim 1**.

$$\begin{aligned} \int_M |\nabla \varphi|_\varphi^2 \omega_\varphi^n &= - \int_M (\varphi - c) \Delta_\varphi \varphi \omega_\varphi^n \\ &= \int_M (\varphi - c) \operatorname{tr}_\varphi \omega_t \cdot \omega_\varphi^n - n \int_M (\varphi - c) \omega_\varphi^n \\ &\leq C. \end{aligned}$$

In last inequality, we have used the fact that  $\varphi$  is uniformly bounded.

**Claim 4:** For any positive  $\chi$ , the following inequality holds

$$\int_0^t \int_M \chi f^2 \operatorname{tr}_\varphi \omega_t \omega_t^n \, d u \leq C_1 t + C_2 \sqrt{t}, \quad (2.9)$$

where both constants depend only on  $|\chi|_{L^\infty}$  and  $|\nabla \chi|_{L^\infty}$ .

**Proof of Claim 4:**

$$\begin{aligned}
& \int_M \chi f^2 \operatorname{tr}_\varphi \omega_t \omega_t^n \\
&= \int_M \chi f \omega_t \wedge \omega_\varphi^{n-1} \\
&= \int_M \chi f \omega_\varphi^n - \int_M \chi f \partial \bar{\partial} \varphi \wedge \omega_\varphi^{n-1} \\
&\leq C + \int_M \chi \partial f \wedge \bar{\partial} \varphi \wedge \omega_\varphi^{n-1} + \int_M f \partial \chi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^{n-1} \\
&\leq C \left( 1 + \left( \int_M |\nabla f|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \left( \int_M |\nabla \varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} + \left( \int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \cdot \left( \int_M f^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \right) \\
&\leq C \left( 1 + \left( \int_M |\nabla f|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} + \left( \int_M f^2 |\nabla \varphi|_\varphi^2 \omega_\varphi^n \right)^{\frac{1}{2}} \right).
\end{aligned}$$

We have used **Claims 2** and **Claim 3** in deriving last inequality..

Then **Claim 4** follows from integrating the above inequality from 0 to  $t$  and using **Claim 1** and the Schwartz inequality.

A straightforward computation shows

$$\begin{aligned}
\frac{\partial \log f}{\partial t} &= \operatorname{tr}_\varphi \left( \frac{\partial}{\partial t} \omega_t \right) + \Delta_\varphi \varphi' - \operatorname{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \\
&= \Delta_\varphi \log f + \operatorname{tr}_\varphi \left( -\partial \bar{\partial} h_{\omega_t} + \frac{\partial}{\partial t} \omega_t \right) - \operatorname{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \\
&\leq \frac{\Delta_\varphi f}{f} - \frac{|\nabla f|_\varphi^2}{f^2} + C \operatorname{tr}_\varphi \omega_t + C.
\end{aligned}$$

Using this, we deduce

$$\begin{aligned}
\frac{d}{dt} \int_M \chi f^2 \omega_t^n &= \int_M \chi \left( 2f \frac{\partial f}{\partial t} + f^2 \operatorname{tr}_{\omega_t} \frac{\partial}{\partial t} \omega_t \right) \omega_t^n \\
&\leq 2 \int_M \chi f \left( \Delta_\varphi f - \frac{|\nabla f|_\varphi^2}{f} + C f \operatorname{tr}_\varphi \omega_t + C f \right) \omega_t^n \\
&\leq -2 \int_M (\nabla \chi, \nabla f)_\varphi \omega_\varphi^n + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \operatorname{tr}_\varphi \omega_t \omega_t^n \\
&\leq 2 \sqrt{\int_M |\nabla \chi|_\varphi^2 \omega_\varphi^n} \sqrt{\int_M |\nabla f|_\varphi^2 \omega_\varphi^n} + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \operatorname{tr}_\varphi \omega_t \omega_t^n \\
&\leq C \sqrt{\int_M |\nabla f|_\varphi^2 \omega_\varphi^n} + C \int_M \chi f^2 \omega_t^n + C \int_M \chi f^2 \operatorname{tr}_\varphi \omega_t \omega_t^n.
\end{aligned}$$

Note that

$$\int_M \chi \cdot f^2 \omega_t^n \leq C, \quad \forall t \in [0, T].$$

Integrating the above inequality from  $t = 0$  to  $t = t_i$  and using **Claim 4**, we have

$$\int_M \chi f^2 \omega_t^n|_{t_i} \leq \int_M \chi f^2 \omega_t^n|_0 + C(t_i + \sqrt{t_i}).$$

In deriving the above inequality, we used the fact that  $\varphi_0$ . For general  $\varphi_0$  as given, applying the above to  $\varphi(s, t_i)$  for any  $s > 0$  and then taking the limit as  $s$  tends to 0, we get for any  $\varphi(t_i)$ ,

$$\int_M \chi \frac{\omega_{\varphi(t_i)}^n}{\omega_t^n} \omega_t^n \leq \int_M \chi \frac{\omega_{\varphi_0}^n}{\omega_t^n} \omega_t^n + C(t_i + \sqrt{t_i}).$$

On the other hand, using the assumption that  $\varphi(t_i)$  converges to  $\tilde{\varphi}_0$ , we can show that  $\frac{\omega_{\varphi(t_i)}^n}{\omega_i^n}$  converges weakly to  $\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n}$  in  $L^2(M, \omega)$ . Then by taking  $t_i$  to 0, we have

$$\int_M \chi \left( \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \right)^2 \omega^n \leq \int_M \chi \left( \frac{\omega_{\varphi_0}^n}{\omega^n} \right)^2 \omega^n. \quad (2.10)$$

Since this holds for any non-negative smooth cut-off function  $\chi$ , we have

$$0 \leq \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \leq \frac{\omega_{\varphi_0}^n}{\omega^n}$$

a.e. in  $M$ . However,

$$\int_M \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \omega^n = \int_M \frac{\omega_{\varphi_0}^n}{\omega^n} \omega^n = \text{vol}(M)!$$

It follows

$$\omega_{\tilde{\varphi}_0}^n \equiv \omega_{\varphi_0}^n \quad (2.11)$$

in  $L^2(M, \omega)$ . The uniqueness of Monge-Ampère equation for  $C^0$  solution by Kolodziej implies that  $\tilde{\varphi}_0 = \varphi_0$ . Since  $\{t_i\}$  is any sequence going to 0, we have proved that  $\varphi(t)$  converges to  $\varphi_0$  as  $t$  tends to 0.

Furthermore, we have

$$\int_M \chi \left( \frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n} \right)^2 \omega^n = \lim_{i \rightarrow \infty} \int_M \chi \left( \frac{\omega_{\varphi_i}^n}{\omega^n} \right)^2 \omega^n.$$

Thus,  $\frac{\omega_{\varphi_i}^n}{\omega^n}$  converges strongly to  $\frac{\omega_{\tilde{\varphi}_0}^n}{\omega^n}$  in  $L^2(M, \omega)$ . Lemma 2.5 is proved. □

So far, we have shown that with assumption on the volume form, the solution of the weak Kähler-Ricci flow really goes to the initial data as  $t \rightarrow 0$ .

### 3 Higher order estimates

In this section, we prove the regularity of the weak Ricci flow under appropriate assumptions on the initial Kähler potential  $\varphi_0$ .

### 3.1 The $C^{1,1}$ case

In this subsection, we want to prove the following Laplacian estimates.

**Proposition 3.1.** *[8] If  $\varphi_0 \in \bar{C}l_{L^\infty}P(M, \omega)$  is  $C^{1,1}$ -bounded, then*

$$0 \leq \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) \leq C, \quad \forall t \in [0, T].$$

*In other words,  $\varphi(t)$  is uniformly  $C^{1,1}$ -bounded. Moreover,  $\frac{\omega_\varphi^n}{\omega^n}$  converges to  $\frac{\omega_{\varphi_0}^n}{\omega^n}$  strongly in  $L^2$  as  $t \rightarrow 0$ .*

*Proof.* In this proof, we always use  $C, C'$  etc. to denote uniform constants, though they can vary at different places. The  $C^{1,1}$  assumption on  $\varphi_0$  gives a  $L^\infty$ -bound for the volume form. In light of estimates from last Section, it suffices to prove the Laplacian estimate.

Let  $\varphi_0(s)$  be the smooth approximations of  $\varphi_0$  and  $\varphi(s, t)$  be the associated solutions as before. Since  $\varphi_0$  is in  $C^{1,1}$ , we can arrange  $\varphi_0(s)$  with uniformly bounded  $C^{1,1}$ -norms. Thus, we only need to bound  $\varphi(s, t)$  uniformly in  $C^{1,1}$ . For simplicity, we just write  $\varphi$  for  $\varphi(s, \cdot)$ .

We first derive a uniform bound on the volume form for all  $t \geq 0$ . By direct computations, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\frac{\partial \varphi}{\partial t}) &\leq \Delta(\frac{\partial \varphi}{\partial t}) + C(1 + \text{tr}_\varphi \omega_t) \\ &\leq \Delta(\frac{\partial \varphi}{\partial t} - C\varphi) + C + C(\text{tr}_\varphi(\partial\bar{\partial}\varphi) + \text{tr}_\varphi \omega_t) \\ &\leq \Delta(\frac{\partial \varphi}{\partial t} - C\varphi) + C + n \\ &\leq \Delta(\frac{\partial \varphi}{\partial t} - C\varphi) + C' + C\varphi. \end{aligned}$$

It can be reformulated as

$$\frac{\partial}{\partial t}(\frac{\partial \varphi}{\partial t} - C\varphi) \leq \Delta(\frac{\partial \varphi}{\partial t} - C\varphi) + C' - C(\frac{\partial \varphi}{\partial t} - C\varphi).$$

Applying the Maximum Principle, one can easily derive a uniform upper bound for  $\frac{\partial \varphi}{\partial t} - C\varphi$ . Since  $\varphi$  is uniformly bounded, we get a uniform upper bound on  $\frac{\partial \varphi}{\partial t}$ , so the volume form is uniformly bounded.

By standard computations following [30], we have

$$e^{C\varphi}(\Delta_{\omega_\varphi} - \frac{\partial}{\partial t})(e^{-C\varphi}\langle \omega_t, \omega_\varphi \rangle) \geq -C + (C\frac{\partial \varphi}{\partial t} - C)\langle \omega_t, \omega_\varphi \rangle + Ce^{-\frac{1}{n-1}\frac{\partial \varphi}{\partial t}}\langle \omega_t, \omega_\varphi \rangle^{\frac{n}{n-1}}.$$

Suppose that the maximum of  $e^{-C\varphi}\langle \omega_t, \omega_\varphi \rangle$  is attained at some  $(p, t)$  in  $M \times [0, T]$ . At that point, we have

$$0 \geq -C + (C\frac{\partial \varphi}{\partial t} - C)\langle \omega_t, \omega_\varphi \rangle + Ce^{-\frac{1}{n-1}\frac{\partial \varphi}{\partial t}}\langle \omega_t, \omega_\varphi \rangle^{\frac{n}{n-1}}.$$

In other words, we have

$$\begin{aligned}\langle \omega_t, \omega_\varphi \rangle^{\frac{n}{n-1}} &\leq C e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} + C e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \langle \omega_t, \omega_\varphi \rangle - C \frac{\partial \varphi}{\partial t} e^{\frac{1}{n-1} \frac{\partial \varphi}{\partial t}} \langle \omega_t, \omega_\varphi \rangle \\ &\leq C + C' \langle \omega_t, \omega_\varphi \rangle.\end{aligned}$$

This implies a uniform upper bound on  $\langle \omega_t, \omega_\varphi \rangle$  at  $(p, t)$ . Since  $\varphi$  is uniformly bounded, it follows that this trace is uniformly bounded. The proposition is thus proved.  $\square$

### 3.2 The $L^\infty$ case

We are going to prove Theorem 1.1 in this subsection. We will derive the Laplacian estimate for  $t > 0$  under the weaker assumption that the initial Kähler potential is only in  $L^\infty$ .

Let's point out that in this case, one can still define weak flow using smooth (decreasing) approximation of the initial bounded Kähler potential (provided in [1]). These flows decrease to the weak flow pointwisely, which guarantees the uniqueness of the weak flow and makes sure that what we are discussed below is the same as before with more regularity assumption..

Recall that  $\omega_t$  is a smooth family of Kähler metrics with  $[\omega_t] = [\omega] - tc_1(M)$  and  $\omega_\varphi = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi$ . Moreover, the Kähler-Ricci flow is reduced to the following equation for  $\varphi$ :

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_t^n} - h_{\omega_t}, \quad \varphi(0, \cdot) = \varphi_0.$$

Because  $\varphi(t)$  is the limit of  $\varphi(s, t)$  as  $s$  tends to 0, in order to prove the theorem, we only need to get uniform estimates on higher order derivatives of  $\varphi(s, t)$  for  $t > 0$ . As before, for simplicity, we can simply assume that  $\varphi_0$  is smooth and  $\varphi(s, \cdot) = \varphi(\cdot)$ .

Choose any subinterval  $[t_1, t_0]$  with  $T > t_1 > t_0 > 0$ . By Lemma 2.2 and 2.3, we have the uniform bound for  $\varphi$  and  $\frac{\partial \varphi}{\partial t}$  for  $t \in [t_0, t_1]$ .

Now we derive the second and higher derivative estimates by the standard methods (see [30], also see [5] or [32]). The estimates may depend on  $t_0$ .

As in last subsection, we have

$$e^{C\varphi} \left( \Delta_{\omega_\varphi} - \frac{\partial}{\partial t} \right) (e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle) \geq -C - C \langle \omega_t, \omega_\varphi \rangle + C \langle \omega_t, \omega_\varphi \rangle^{\frac{n}{n-1}},$$

where the bound on  $\frac{\partial \varphi}{\partial t}$  for  $t \in [t_0, t_1]$  has been used. Note that  $\langle \omega_t, \omega_\varphi \rangle$  is nothing but  $n + \Delta_t \varphi$ .

Since  $e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle$  may not be bounded at  $t = 0$ , we consider  $(t - t_0)^{n-1} e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle$  instead. As in last subsection, by the standard computations, we can have

$$\begin{aligned}& e^{C\varphi} \left( \Delta_{\omega_\varphi} - \frac{\partial}{\partial t} \right) ((t - t_0)^{n-1} e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle) \\ & \geq (t - t_0)^{n-1} (-C - C \langle \omega_t, \omega_\varphi \rangle + C \langle \omega_t, \omega_\varphi \rangle^{\frac{n}{n-1}}) - (n-1)(t - t_0)^{n-2} \langle \omega_t, \omega_\varphi \rangle.\end{aligned}\quad (3.1)$$

At the maximum value point  $(p, \tilde{t})$  of  $(t - t_0)^{n-1} e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle$  in  $M \times [t_0, t_1]$ , clearly,  $\tilde{t} > t_0$ . At the point  $(p, \tilde{t})$ ,

$$\begin{aligned} 0 &\geq (\tilde{t} - t_0)^{n-1} (-C - C \langle \omega_{\tilde{t}}, \omega_\varphi \rangle + C \langle \omega_{\tilde{t}}, \omega_\varphi \rangle^{\frac{n}{n-1}}) - (n-1)(\tilde{t} - t_0)^{n-2} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle \\ &= -C(\tilde{t} - t_0)^{n-1} - (C\tilde{t} + C)(\tilde{t} - t_0)^{n-2} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle + C(\tilde{t} - t_0)^{n-1} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle^{\frac{n}{n-1}}. \end{aligned}$$

Multiple  $\tilde{t} - t_0$  on both sides and reformulate the above to be

$$0 \geq -C(\tilde{t} - t_0)^n - (C\tilde{t} + C)(\tilde{t} - t_0)^{n-1} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle + C((\tilde{t} - t_0)^{n-1} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle)^{\frac{n}{n-1}}.$$

So we get

$$(\tilde{t} - t_0)^{n-1} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle \leq C.$$

Thus at  $(p, \tilde{t})$ , using the bound for  $\varphi$ ,

$$(\tilde{t} - t_0)^{n-1} e^{-C\varphi} \langle \omega_{\tilde{t}}, \omega_\varphi \rangle \leq C'.$$

It follows that for all  $t \in [t_0, t_1]$ ,

$$(t - t_0)^{n-1} e^{-C\varphi} \langle \omega_t, \omega_\varphi \rangle \leq C', \quad \text{on } M$$

which implies

$$\langle \omega_t, \omega_\varphi \rangle \leq \frac{C}{(t - t_0)^{n-1}}.$$

Next, we can get the third order estimate in a similar fashion. We have a similar inequality as in [5]. Of course, in order to do this, we still translate the time to guarantee the uniform metric bound. Let's say the time is translated so that the original time  $t = t_0 > 0$  is now the new initial time  $t = 0$ .

The inequality is as follows, with  $S = g_\varphi^{i\bar{j}} g_\varphi^{k\bar{l}} g_\varphi^{\lambda\bar{\eta}} \varphi_{i\bar{l}\lambda} \varphi_{j\bar{k}\eta}$ ,

$$(\Delta_\varphi - \frac{\partial}{\partial t})S \geq -C \cdot S - C.$$

Recalled from previous subsection, we also have

$$(\Delta_\varphi - \frac{\partial}{\partial t})\Delta_{\omega_t}\varphi \geq C \cdot S - C.$$

From the first one, as in the Laplacian estimate, we use,

$$(\Delta_\varphi - \frac{\partial}{\partial t})(tS) \geq -Ct \cdot S - Ct - S.$$

By choosing  $A > 0$  large enough, one has, for  $t \in [0, T]$ ,

$$(\Delta_\varphi - \frac{\partial}{\partial t})(tS + A\Delta_{\omega_t}\varphi) \geq C \cdot S - C.$$

The function  $tS + A\Delta_{\omega_t}\varphi$  is uniformly bounded at the (new) initial time,  $t = 0$ . By the Maximum Principle, if its maximum value is taken at some point in  $M$  and  $t > 0$ , then  $S$  is bounded there. Consequently, the whole function  $tS + A\Delta_{\omega_t}\varphi$  is also bounded there. So we finally conclude

$$tS + A\Delta_{\omega_t}\varphi \leq C,$$

which implies

$$S \leq \frac{C}{t}, \quad 0 < t \leq T.$$

This provides a local  $C^{2,\alpha}$  bound for  $\varphi$ . After this, higher order estimates follow from the standard parabolic version of Schauder Estimates.

Therefore, we have proved that the weak flow defined before is actually smooth in  $(0, T]$ . Theorem 1.1 follows from this and the second order estimate in section 3.1. Theorem 1.3 follows from this and last part of Section 2.

## 4 $C^{1,1}$ K-energy minimizer

In an earlier paper [8] where the present work is initiated, the first two named authors proved that

**Proposition 4.1.** *In any Kähler class, the volume form of any  $C^{1,1}$  K-energy minimizer belongs to  $H^{1,2}(M, \omega)$ .*

By Lemma 2.8 and Theorem 3.3, we can prove a stronger theorem.

**Theorem 4.2.** *The  $C^{1,1}$  minimizer of the K-energy functional in any canonical Kähler class with positive first Chern class is necessarily smooth.*

*Proof.* Let  $\phi_0$  be a  $C^{1,1}$  Kähler potential which minimizes the K-energy functional in the canonical Kähler class. According to the results proved in the preceding sections, there is a unique smooth Kähler-Ricci flow  $\varphi(t)$  ( $t > 0$ ) such that

$$\frac{\partial}{\partial t}\varphi = \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi - h_{\omega}.$$

Moreover,  $\lim_{t \rightarrow 0} \frac{\omega_{\varphi}^n}{\omega^n} = \frac{\omega_{\phi_0}^n}{\omega^n}$  strongly in  $L^2(M, g_0)$  and the Kähler potential  $\varphi(t)$  converges to  $\phi_0$  strongly in  $C^{1,\alpha}(M)$  for any  $\alpha \in (0, 1)$ . Since the K-energy is decreasing along the Kähler-Ricci flow, we have

$$\limsup_{t \rightarrow 0^+} \mathbf{E}(\varphi(t)) \leq \mathbf{E}(\phi_0) = \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi).$$

Since the K-energy is non-increasing for  $\varphi(t)$  ( $t > 0$ ), then for any  $t > 0$ , we have

$$\inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi) \leq \mathbf{E}(\varphi(t)) \leq \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi).$$

In other words,

$$\mathbf{E}(\varphi(t)) = \inf_{\phi \in \mathcal{H}} \mathbf{E}(\phi), \quad t > 0.$$

Since  $\omega_{\varphi(t)}$  is a smooth Kähler metric, this means that the scalar curvature of  $\omega_{\varphi(t)}$  ( $t > 0$ ) must be constant. Consequently,  $\omega_{\varphi(t)}$  is a Kähler-Einstein metric for all  $t > 0$ . This in turns implies that  $\frac{\partial \varphi}{\partial t}$  is a functional of  $t$  only. Note that  $\varphi_0$  is the strong  $C^{1,\alpha}$  limit of  $\varphi(t)$  as  $t \rightarrow 0$ . Therefore,  $\varphi_0 - \varphi(t)$  is a constant which depends only on  $t$ . In other words,  $\omega_{\varphi_0}$  is also a Kähler-Einstein metric and the theorem is then proved.  $\square$

## References

- [1] B., Zbigniew; K., Slawomir. On regularization of plurisubharmonic functions on manifolds. Proc. Amer. Math. Soc. 135 (2007), no. 7, 2089–2093 (electronic).
- [2] S. Bando and T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
- [3] E. Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 16 of 102, pages 259–290. Ann. of Math. Studies, University Press, 1982.
- [4] E. Calabi. Extremal Kähler metrics, II. In *Differential geometry and Complex analysis*, pages 96–114. Springer, 1985.
- [5] H. D. Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. Invent. Math. 81 (1985), no. 2, 359–372.
- [6] X. X. Chen. Weak limits of Riemannian metrics in surfaces with integral curvature bound. Calc. Var. Partial Differential Equations **6** (1998), no. 3, 189–226.
- [7] X. X. Chen. Space of Kähler metrics. *Journal of Differential Geometry*, 56(2):189–234, 2000.
- [8] X. X. Chen and G. Tian. Foliation by holomorphic discs and its application in Kähler geometry, math.DG/0507148. Submitted.
- [9] X. X. Chen. Space of Kähler metrics (III)—On the greatest lower bound of the Calabi energy and the lower bound of geodesic distance. Preprint, 2006.



- [10] X. X. Chen and W. Y. Ding. Ricci flow on surfaces with  $L^\infty$  initial metrics. Ricci flow on surfaces with degenerate initial metrics. *J. Partial Differential Equations* 20 (2007), no. 3, 193–202.
- [11] X. X. Chen and G. Tian. Ricci flow on Kähler-Einstein surfaces. *Invent. Math.* 147 (2002), no. 3, 487–544.
- [12] X. X. Chen and G. Tian. Ricci flow on Kähler-Einstein manifolds. *Duke. Math. J.* 131, (2006), no. 1, 17–73.
- [13] B. Chow. The Ricci flow on the 2-sphere. *J. Differential Geom.* 33 (1991), no. 2, 325–334.
- [14] B. Chow, P. Lu and L. Ni Hamilton’s Ricci flow. Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006. xxxvi+608 pp. ISBN: 978-0-8218-4231-7; 0-8218-4231-5
- [15] W.Y. Ding and G. Tian. Kähler-Einstein metrics and the generalized Futaki invariant. *Invent. Math.* 110 (1992), no. 2, 315–335.
- [16] S. K. Donaldson. Scalar curvature and projective embeddings, II. *Q. J. Math.* 56 (2005), no. 3, 345–356.
- [17] S. K. Donaldson. Lower bounds on the Calabi functional. *math.DG/0506501*.
- [18] R. Hamilton. Three-manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.
- [19] R. Hamilton. *The formation of singularities in the Ricci flow*, volume II. Internat. Press, 1993.
- [20] G. Huisken. Ricci deformation of the metric on a Riemannian manifold. *J. Differential Geom.* 21 (1985), no. 1, 47–62.
- [21] T. Mabuchi. K-energy maps integrating Futaki invariants. *Tohoku Math. J.*, **38** (1986), 575–593.
- [22] T. Mabuchi An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I. *Invent. Math.* 159 (2005), no. 2, 225–243.
- [23] G. Perelman. The entropy formula for the Ricci flow and its geometric applications, <http://arxiv.org/abs/math.DG/0211159>.
- [24] K., Slawomir. The complex Monge-Ampere equation and pluripotential theory. *Mem.Amer. Math. Soc.* 178 (2005), no. 840, x+64 pp.

- [25] K., Slawomir. Hölder continuity of solutions to the complex Monge-Ampère equation with the right hand side in  $L^p$ . The case of compact Kähler manifolds. ArXiv, math.CV/0611051.
- [26] J. Song and G. Tian. The Kähler-Ricci flow on surfaces of positive Kodaira dimension/ *Inventiones math.*, 170(2007), no. 3, 609-653.
- [27] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.* 130 (1997), no. 1, 1–37.
- [28] G. Tian and X. H. Zhu Convergence of Kähler-Ricci flow. *J. Amer. Math. Soc.* 20 (2007), no. 3, 675–699.
- [29] G. Tian and Z. Zhang. On the Kähler-Ricci flow on projective manifolds of general type. *Chinese Annals of Mathematics - Series B*, Volume 27, Number 2, 179–192.
- [30] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math.* 31 (1978), no. 3, 339–411.
- [31] Z. Zhang. On Degenerate Monge-Ampere Equations over Closed Kähler Manifolds. *Int. Math. Res. Not.* 2006, Art. ID 63640, 18 pp.
- [32] Z. Zhang. Degenerate Monge-Ampere Equations over Projective Manifolds. PHD Thesis at MIT, 2006.

X. X. Chen, Department of Mathematics, University of Wisconsin, Madison, WI 53706/  
xiu@math.wisc.edu

G. Tian, Department of Mathematics, Princeton University, Princeton, NJ 08544/  
tian@math.princeton.edu

Z. Zhang, Department of Mathematics, University of Michigan, at Ann Arbor, MI 48109/  
zhangou@umich.edu